MATH 239: Intro to Combinatorics **Fall 2024**
Formula / Theorem Sheet **by Strafe < 3**

1 Week 1

1.1 Choices of AND / OR

On a table there are 7 apples, 8 oranges, 5 bananas
An apple **and** a banana: $7 \times 5 = 35$
An apple **or** an orange: $7 + 8 = 15$

1.2 Lists and Permutation

A set of S is a list of elements of S exactly one of each. For example, $\{1, a, X, g\}$ are

$$1aXg, a1Xg, X1ag, g1aX, \dots$$

A permutation is a list of the set $\{1, 2, \dots, n\}$
Theorem For every $n \geq 1$, the number of lists of an n-element set S is

$$n(n-1)(n-2)\dots 3 \cdot 2 \cdot 1 = n!$$

1.3 Number of Subsets

For every $n \geq 0$, the number of subsets of an $n - element$ set is 2^n.
A **partial** list of a set S is a list of subset of S.

1.4 Number of Partial Lists

The number of partial lists of length k of an n-element set is $n(n-1)\dots(n-k+2)(n-k+1)$

1.5 Number of k-element subsets

For $0 \leq k \leq n$ the number of k-element subsets of an n-element set is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

1.6 Multisets

Let $n \geq 0$ and $t \geq 1$ be integers. A **multiset** of size n with elements of t types is a sequence of nonnegative integers $(m_1, \dots, m_t)$ s.t.

$$m_1 + m_2 + \cdots + m_t = n$$

The number of $n - element$ mutlisets with elements of t types is

$$\binom{n+t-1}{t-1}$$

2 Week 2

2.1 Bijection

Let A and B be sets and let $f : A \to B$

- f is **surjective (onto)** if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$
- f is **injective (one-to-one)** if for every $a, a' \in A$, if $f(a) = f(a')$ then $a = a'$
- f is **bijective** if its both surjective and injective.

Corollary: If there exists a bijection between two sets A and B and at least one is finite, they are both finite and $|A| = |B|$
Proposition: Let $f : A \to B$ and $g : B \to A$ be functions between two sets A and B. Assume the following

- For all $a \in A$, $g(f(a)) = a$
- For all $b \in B$, $f(g(b)) = b$

Then both f, g are bijections. Moreover, $a \in A, b \in B$, we have $f(a) = b \iff g(b) = a$

2.2 Formal Power Series

It is an expression of the form

$$G(x) = \sum_{n \geq 0} g_n x^n$$

where the coefficients $(g_0, g_1, g_2, \dots)$ are a sequence of real numbers.
Proposition: The inverese of $F(x) = \sum_{n \geq 0} f_n x^n$ exists if and only if $f_0 \neq 0$

2.3 Binomial Theorem

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k \geq 0} \binom{n}{k} x^k$$

2.4 Negative Binomial Theorem

$$(1 - x)^{-t} = \sum_{n \geq 0} \binom{n + t - 1}{t - 1} x^n$$

3 Week 3

3.1 Coefficient Extraction

Given $G(x) = g_1 x + g_2 x^2 + \cdots + g_n x^n$

$$[x^k]G(x) = g_k$$

Some Rules:

1. $[x^k]aF(x) + bG(x) = a[x^k]F(x) + b[x^k]G(x)$
2. $[x^k](x^\ell F(x)) = [x^{k-\ell}]F(x)$
3. $[x^k]F(x)G(x) = \sum_{\ell=0}^{k}([x^\ell]F(X))([x^{k-\ell}]G(x))$

3.2 Example of Generating Series

Given $\mathcal{M} = \{$ Jan, Feb, ..., Dec $\}$

$$\mathcal{M}_n = \{\alpha \in \mathcal{M}, \alpha \text{ has exactly n days}\}, \mathcal{M} = \bigcup_{n\geq 0} \mathcal{M}_n$$

$$\mathcal{M}_0 = \varnothing, \mathcal{M}_{28} = \{\text{Feb}\}, \mathcal{M}_{30} = \{\text{April, June, Sept, Nov }\}, \ldots$$

3.3 Weight Function

Let A be a set. A function $\omega : A \to \mathbb{N}$ from A to set $\mathbb{N}$ of natural numbers is a **weight function** provided that all of $n \in \mathbb{N}$, the set

$$A_n = \omega^{-1}(n) = \{a \in A : \omega(a) = n\}$$

is **finite**. That is for every $n \in \mathbb{N}$ there are only finitely many elements $a \in A$ of weight n.
Proposition:

$$\Phi_A(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

For every $n \in \mathbb{N}$, the number of elements of A of weight n is $a_n = |A_n|$
Proposition:

$$\Phi_S(x) = \sum_{n\geq 0} |\{\alpha \in S : \omega(\alpha) = n| \cdot x^n$$

3.4 Sum and product Lemmas

Let S_1 be disjoint sets and ω be a weight function of $S_1 \cup S_2$

$$\Phi_{S_1}(x) + \Phi_{S_2}(x) = \Phi_{S_1 \cup S_2}(x)$$

Let $S_0, S_1, S_2, \ldots$ be disjoint sets with union S and let ω be a weight function of S.

$$\Phi_S(x) = \sum_{n \geq 0} \Phi_{S_n}(x)$$

Let S_1, S_2 be sets and let ω_1, ω_2 be associated weight functions.

$$\Phi^{\omega_1}_{S_1}(x)\Phi^{\omega_2}_{S_2}(x) = \Phi^{\omega}_{S_1 \times S_2}(x)$$

where ω is a weight function on $S_1 \times S_2$ defined by $\omega(\alpha_1, \alpha_2) = \omega(\alpha_1) + \omega(\alpha_2)$
Lemma: Let A be a set with weight function: $w : A \to \mathbb{N}$ and define A^* and $w^* : A^* \to \mathbb{N}$ as above. Then w^* is a weight function on A^* if and only if there are no elements in A of weight zero (that is $A_0 = \varnothing$)

3.5 Star operator

$$A^* = \cup_{k \geq 0} A^k = \{\text{all tuples of elements of } A\}$$

For example, $$\{0,1\}^* = \{\underbrace{\overbrace{()}^{\varepsilon}}_{A^0}, \underbrace{\overbrace{(0)}^{0}, \overbrace{(1)}^{1}}_{A^1}, \underbrace{\overbrace{(0,0)}^{00}, \overbrace{(0,1)}^{01}, \overbrace{(1,0)}^{10}, \overbrace{(1,1)}^{11}}_{A^2}, \underbrace{\overbrace{(0,0,0)}^{000}, \ldots}_{A^3, \ldots}\}$$

3.6 String Lemma

Let A be a set with weight function ω such that no elements of A have weight 0. Then

$$\Phi^{w^*}_{A^*}(x) = \frac{1}{1 - \Phi^{\omega}_{A}(x)}$$

3.7 Composition

A composition is a finite sequence of positive integers

$$\gamma = (c_1, c_2, \ldots, c_k)$$

Each $c_i \in \mathbb{Z}_{>0}$ is called a **part**. The **length** of the composition is the number of parts $\ell(\gamma) = k$. The size of the composition is the sum of parts, $|\gamma| = c_1 + c_2 + \cdots + c_k$. If s is the size of γ then we say γ is a composition of s.

3.8 Composition Theorem

Let $P = \{1, 2, 3, \dots\}$ be positive integers

- The set C of all combinations is $C = P^*$
- The generating series for all integers compositions with respect to size is

$$\Phi_{\text{compositions}}(x) = \frac{1-x}{1-2x} = 1 + \frac{x}{1-2x}$$

- For $n \in \mathbb{N}$ the number of compositions of size n is

$$|C_n| = \begin{cases} 1 & \text{if } n = 0, \\ 2^{n-1} & \text{if } n \geq 1 \end{cases}$$

4 Week 4

4.1 Binary String

A **binary string** of length $n \geq 0$ is a finite sequence $\sigma = b_1 b_2 \ldots b_n$ where each bit $b_i \in \{0, 1\}$. A binary string of length n is an element of the Cartesian power $\{0, 1\}^n$. A binary string of arbitrary length of the set $\{0, 1\}^* = \cup_{n=0}^{\infty} \{0, 1\}^n$. One binary string ϵ of length zero, empty string with no bits.

4.2 Concatenation Product

If S and T are sets of binary strings, then

$$ST = \{\sigma \mathcal{T} : \sigma \in S, \mathcal{T} \in T\}$$

and $S^k = SS \ldots S$

4.3 Regular Expression

A **regular expression** is defined recursively as any of the following

- $\epsilon, 0$, or 1
- the expression $R \smile S$ where R and S are regular expressions
- the expression RS where R and S are regular expressions with $R^k = RR \ldots R$ for any $k \in \mathbb{N}$
- the expression R^* where R is a regular expression

4.4 Rational languages

We define production recursively

- The regular expression $\epsilon, 0, 1$ produces these sets $\{\epsilon\}, \{0\}, \{1\}$ respectively.
- If R produces $\mathcal{R}$ and S produces $\mathcal{S}$ then
  - $R \smile S$ produces $\mathcal{R} \cup \mathcal{S}$ (set union)
  - RS produces $\mathcal{RS}$ (concatenation product)
  - R^* produces $\mathcal{R}^* = \cup_{k \geq 0} \mathcal{R}^k$ (concatenation powers)

4.5 Unambiguous expressions

A regular expression R is unambiguous if every string in the langauge $\mathcal{R}$ produced by R is produced in exactly one way by R. Otherwise, R is ambiguous.
Lemma:

- The regular expression $\epsilon, 0, 1$ are unambiguous expressions

- If R and S are unambiguous expressions that produces $\mathcal{R}$ and $\mathcal{S}$ respectively, then
  - $R \smile S$ is unambiguous iff $R \cap S = \varnothing$ (disjoint)
  - RS is unambiguous iff there is a bijection between RS and $R \times S$. In other words, for every string $\alpha \in RS$, there is exactly one way to write $\alpha = \rho\sigma$ with $\rho \in \mathcal{R}$ and $\sigma \in \mathcal{S}$
  - R^* is unambiguous if and only if each of the concatenation product R^k is unambiguous and all of the R^k are disjoint

5 Week 5

5.1 Regular Expression and Rational Functions

A regular expression leads to a rational function, defined recursively as follows

- Regular expressions $\epsilon, 0, 1$ leads to formal power series $1, x, x$
- If R, S are regex that leads to $f(x), g(x)$ then
  - $R \smile S$ leads to $f(x) + g(x)$
  - RS leads to $f(x) \cdot g(x)$
  - R^* leads to $\dfrac{1}{1 - f(x)}$

Theorem:
Let R be a regular expressions that unambiguously produces the language $\mathcal{R}$. Also suppose that $\mathcal{R}$ leads to $f(x)$. Then the generating series for $\mathcal{R}$ with respect to length is $f(x)$ i.e. $\Phi_{\mathcal{R}}(x) = f(x)$

5.2 Block of a string

A **block** of a binary string s is a nonempty maximal substring of equal bits
Proposition:
The regular expression (0*)(11*00*)*1* is unambiguous and produces the set of all binary strings. Same for 1*(00*11*)*0*

5.3 Pre/postfix decompositions

A **prefix decomposition** has the form A^*B. A **postfix decomposition** has the form AB^*

5.4 Recursive decomposition

A **recursive decomposion** of a set $\mathcal{S}$ describes S in terms of itself using the language of regular expressions together with the symbol S which produces set $\mathcal{S}$. A recursive decomposition for S is **unambiguous** if each side of the equation produces each string at most once.

5.5 Theorem 3.26

Let $\kappa \in \{0,1\}^*$ be a non empty string of length n and let $A = A_\kappa$ be the set of binary strings that avoid κ. Let C be the set of all nonempty suffixes of γ of κ such that $\kappa\gamma = n\kappa$ for some nonempty prefix n of κ. Let $C(x) = \sum_{\gamma \in C} x^{\ell(\kappa)}$ Then

$$A(x) = \frac{1 + C(x)}{(1 - 2x)(1 + C(x)) = x^n}$$

5.6 Excluded substrings

Check course notes

6 Week 6

6.1 Homogeneous Linear Recurrence Relations

Let $g = (g_0, g_1, g_2, \dots)$ be an infinite sequence of complex numbers. Let $a_1, a_2, \dots, a_d$ be in $\mathbb{C}$, let $N \geq d$ be an integer. We say that g satisfies a **homogeneous linear recurrence relation** provided that

$$g_n + a_1 g_{n-1} + a_2 g_{n-2} + \dots a_d g_{n-d} = 0$$

for all $n \geq N$. The values $g_0, g_1, \dots, g_{N-1}$ are the initial conditions of the currence. The relation is **linear** because LHS is a linear combination of entries of the sequence g. It is **homogeneous** because the **RHS** of equation is zero.

6.2 Theorem 4.8

Let $g = (g_0, g_1, g_2, \dots)$ be a sequence of complex numbers and let $G(x) = \sum_{n=0}^{\infty} g_n x^n$ be the corresponding generating series. The following are equivalent

- The sequence g satisfies a homogeneous linear recurrence relation

$$g_n + a_1 g_{n-1} + \dots + a_d g_{n-d} = 0$$

  for all $n \geq N$ with IC $g_0, g_1, \dots, g_{N-1}$
- The series $G(x) = P(x)/Q(x)$ is a quotient of two polynomials. The demoniator is

$$Q(x) = 1 + a_1 x + a_2 x^2 + \dots a_d x^d$$

  and the numerator is $P(x0 = b_0 + b_1 x + b_2 x^2 + \dots + b_{N-1} x^{N-1}$ in which

$$b_k = g_k + a_1 g_{k-1} + \dots + a_d g_{k-d}$$

  for all $0 \leq k \leq N-1$ with the convention that $g_n = 0$ for all $n < 0$

6.3 Partial fractions (Simple version)

Let that

$$G(x) = \frac{P(x)}{(1-\lambda_1 x)(1-\lambda_2 x)\dots(1-\lambda_s x)}$$

where P is a polynomial of degree less than s, $\lambda_i \in \mathbb{C}$ are distinct. Then there exists $C_1, C_2, \dots C_s \in \mathbb{C}$ s.t.

$$G(x) = \frac{C_1}{1-\lambda_1 x} + \frac{C_2}{1-\lambda_2 x} \dots + \frac{C_s}{1-\lambda_s x}$$

To find C_i, cross-multiply and equate coefficients.

6.4 More Partial Fractions

$$G(x) = \frac{P(x)}{Q(x)} = \frac{P(x)}{1-\lambda_1 x)^{d_1}(1-\lambda_2 x)^{d_2}\dots(1-\lambda_s x)^{d_s}}$$

where $deg(P) < deg(Q)$, the $\lambda_i \in \mathbb{C}$ are distinct and $d_i \geq 1$. Then there exists $C_1^{(1)}, C_1^{(2)}, \dots, C_1^{d_1}, C_2^{(1)}, \dots, C_2^{(2)}$, $\dots, C_s^{(1)}, \dots, C_s^{d_s} \in \mathbb{C}$ s.t.

$$G(x) = \sum_{i=1}^{s}\sum_{j=1}^{d_i} \frac{C_i^{(j)}}{(1-\lambda_i x)^j}$$

6.5 Main Theorem

Let $g = (g_0, g_1, g_2)$ be a sequence of complex numbers and $G(x) = \sum_{n=0}^{\infty} g_n x^n$ be the corresponding generating series. Assume that the equivlanet condiitons of Theorem 4.8 hold and that

$$G(x) = R(x) + \frac{P(x)}{Q(x)}$$

for some polynomial $P(x), Q(x), R(x)$ with $\deg(P(x)) < \deg(Q(X))$ and $Q(0) = 1$. Factor $Q(x)$ to obtain its inverse roots and their multiplicities

$$Q(x) = (1-\lambda_1 x)^{d_1}(1-\lambda_2 x)^{d_2}\dots(1-\lambda_s)^{d_s}$$

Then there are polynomials $p_i(n)$ for $1 \leq i \leq s$ with deg $p_i(n) < d_i$ such that for all $n > degR(x)$

$$g_n = p_1(n)\lambda_1^n + p_2(n)\lambda_2^n + \cdots + p_s(n)\lambda_s^n$$

6.6 Theorem 4.18

Let $g = (g_0, g_1, \dots)$ be a sequence of complex numbers. The following are equivalent,

- The sequence g satisfies a homogeneous linear recurrence relation (with IC)
- The sequence g satisfies a possibly inhomogeneous linear recurrence relation (with IC) in which the RHS is an eventually polyexp function
- The generating series $G(x) = \sum_{n=0} g_n x^n$ is a rational function (a quotient of polynomials in x)
- The sequence $g = (g_0, g_1, g_2, \dots)$ is an eventually polyexp function

6.7 Generating Series Theorem

Let $G(x) = \sum_{n\geq 0} g_n x^n$ be a generating series. The following are equivalent

1. The sequence $g_0, g_1, \dots$ satisfies the homogeneous linear recurrence relation

$$g_n + a_1 g_{n-1} + \cdots + a_d g_{n-d} = 0$$

   for all $n \geq N$ with initial conditions $g_0, g_1, \dots, G_{N-1}$

2. $G(x) = P(x)/Q(x)$ where

$$P(x) = b_0 + b_1 x + b_2 x^2 + \dots b_{N-1} x^{N-1}$$
$$Q(x) = \underbrace{1 + a_1 x + a_2 x^2 + \dots + a_d x^d}_{\text{auxiliary polynomial}}$$

and

$$b_k = g_k + a_1 g_{k-1} + \dots + a_d g_{k-d}$$

for all $0 \le k \le N-1$ with the convention $g_n = 0$ for all $n < 0$

6.8 Recurrence Relation to Explicit Formula Theorem

Suppose $g_0, g_1, \dots$ is a sequence satisfying recurrence relation

$$g_n + a_1 g_{n-1} + \dots + a_d g_{n-d} = 0$$

Factor the auxiliary polynomial to obtain the "inverse roots" $\lambda_i \in \mathbb{C}$ (distinct) and their multiplicities

$$\underbrace{1 + a_1 x + a_2 x^2 + \dots + a_d x^d}_{\text{auxiliary polynomial}} = (1 - \lambda_1 x)^{d_1} (1 - \lambda_2 x)^{d_2} \dots (1 - \lambda_s x)^{d_s}$$

Then

$$g_n = p_1(n)\lambda_1^n + p_2(n)\lambda_2^n + \dots + p_s(n)\lambda_s^n$$

where each p_i is a polynomial of degree less than d_i

6.9 Graph

A graph consists of a finite non-empty set $V(G)$ of objects called vertices and a set of $E(G)$ of edges which are unordered pairs of distinct verticies.
Terminologies:

- Two verticies u, v are **adjacent** if $\{u, v\}$ is an edge
- If $e = \{u, v\}$ is an edge, edge e is an **incident** with vertices u, v
- The vertices adjacent to vertex v are called neighbours of v
- Set of neighbor of v is denoted $N(v)$

We can use $e = uv$ to represent an edge $e = \{u, v\}$. Edges are unordered (undirected) so $e = uv = vu$

7 Week 7

7.1 Isomorephic

Two graphics G_1, G_2 are **isomorphic** if there exists a bijection $f : V(G_1) \to V(G_2)$ such that u, v are adjacent in G_1 iff $f(u), f(v)$ are adjacent in G_2.

7.2 Degree

The degree of a vertex v in a graph G is the number of edges incident with v and is denoted as $deg(v), deg_G(v)$. Degree is also the size of the neighbours $deg(v) = |N(v)|$

7.3 Handshake Lemma

For every graph G, we have

$$\sum_{v \in G} deg(v) = 2|E(G)|$$

Corollary The number of vertices of odd degree in a graph is even
Corollary The average degree of vertex in the graph G is

$$\frac{2|E(G)|}{|V(G)|}$$

A graph in which every vertex has degree k is called a k - **regular graph**.
The number of edges is $nk/2$

7.4 Complete Graph

A **complete graph** is one in which all pairs of distinct vertices are adjacent. It's degree is $|V(G)| - 1$. The complete graph on n vertices is denoted K_n.
The number of edges is $n(n-1)/2 = \binom{n}{2}$

8 Week 8

8.1 Bipartite Graph

A graph is **bipartite** if its vertex set can be partitoned into two disjoint sets A, B such that $B = A \cup B$ and every edge in G has one endpoint in A and one endpoint in B.

8.2 Complete Bipartite Graph

For positive integers m, n, the complete bipartite graph $K_{m,n}$ in the graph with bipartition A, B where $|A| = m, |B| = n$, containing all possible edges joining vertices in A with vertices in B.

8.3 n-cube (Hypercube)

For $n \geq 0$, the n-cube is the graph whose vertex set contains of all binary string of length n and two vertices (*strings*) are adjacent if and only if they differ in exactly one position. Characteristics:

- Number of Vertices: 2^n
- n-regular (can get neighbour of s by changing one position of s)
- Number of edges $= n2^{n-1}$
- It is bipartite

8.4 Adjaceny Matrix

The **adjacency matrix** of a graph with vertices $\{v_1, \dots, v_n\}$ is the $n \times n$ matrix A where

$$A_{i,j} = \begin{cases} 1 & \text{if } v_i, v_j \text{ are adjacent} \\ 0 & \text{o.w.} \end{cases}$$

For simple graphs, A is symmetric and its diagonal is 0.

8.5 Incidence Matrix

The incidence matrix of a graph with vertices $\{v_1, \dots, v_n\}$ and edges $\{e_1, \dots, e_m\}$ is the $n \times n$ matrix B where

$$B_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ are incident with } e_j \\ 0 & \text{o.w.} \end{cases}$$

Each column column contains exactly two 1's (connecting two nodes) and each row sums to the degree of the vertex.

8.6 Subgraph

A graph H is a subgraph of a graph G if the vertex set of H is a non-empty subset of vertex set of $G(V(H) \subseteq V(G))$ and the edge of H is a subset of edge set $G(E(H) \subseteq E(G))$ where both endpoints of any edge in $E(H)$ are in $V(H)$. If $V(H) = V(G)$, we call H a spanning graph. If $H \neq G$, H is a **proper** subgraph.

8.7 Walk and Path

A u, v-**walk** is a sequence of alternating vertices and edges $v_0, v_0v_1, v_1, v_1v_2, v_2, \ldots, v_{k-1}, v_{k-1}v_k, v_k$ where $u = v_0, v = v_k$. The walk has length k. Such a walk is closed if $v_0 = v_k$.
A u, v-**path** is a u, v-walk with no repeated vertices. We can have the trivial empty path (walk length 0)
Theorem: If there is a u, v-walk in G, then there is a u, v-path in G.
Corollary: If there is a u, v-path and a v, w-path in G, then there is a u, w-path in G.

8.8 Cycle

A **cycle** of length $n \geq 1$ is a subgraph with n distinct vertices $v_0, v_1, \ldots, v_{n-1}$ and n distinct edges $v_0v_1, v_1v_2, \ldots, v_{n-1}v_0$
Theorem: If every vertex in G has degree of at least 2, then G contains a cycle.
Hamiltonian Cycle: A cycle that is a spanning subgraph (visits all vertices)

8.9 Girth

The girth of a graph is the length of the shortest cycle. If G has no cycle, the girth is ∞.

8.10 Connected

A graph is **connected** if there exists a u, v-path for every pair of vertices u, v.
Theorem: Let G be a graph and let $u \in V(G)$. Then G is connected if and only if a u, v-path exists for every $v \in V(G)$

8.11 Component

A component C of a graph is a connected subgraph of G such that C is not a proper subgraph of any other connected subgraph of G.

- A disconnected graph has at least 2 components
- A graph with exactly 1 component is connected
- A connected graph has exactly 1 component
- There are no edges joining a vertex of a component with a vertex outside that component (otherwise it is not a maximal connected subgraph)

9 Week 9

9.1 Cut Induced

Let $X \subset V(G)$. The cut induced by X in G is the set if all edges in G with exactly one end in X.
Theorem: A graph G is disconnected iff there exists a non-empty property subset of X of $V(G)$ such that the cut induced by X is empty.

9.2 Eulerian Circuits

An eulerian circuit (Euler tour) of a graph G is a closed walk that contains every edge of G exactly once. Properties:

- Not necessarily connected
- Every vertex must have even degree. Each time we visit a vertex, we must go in and out using distinct edges.

Theorem: Let G be a connected graph. Then G has an eulerian circuit if and only if every vertex has even degree.

9.3 Bridge

A bridge (cut-edge) is an edge e of G if $G - e$ has more components than G.
Lemma: If $e = xy$ is a bridge in a conneted graph G, then $G - e$ has exactly two components, and x and y are in different components of $G - e$.
Theorem: An edge e is a bridge of G iff it is not contained in any cycle of G.
Corollary: If there are two distinct paths from $u \to v$ in G, then G contains a cycle.

9.4 Tree

A **tree** is a connected graph with no cycles.
A **forest** is a graph with no cycles.
Lemma: Every edge in a tree or forest is a bridge
Lemma: Let x, y be vertices in a tree T. Then there exists a unique x, y-path in T.
Theorem: If T is a tree then $|E(T)| = |V(T)| - 1$

9.5 Leaf

A **leaf** is a vertex of degree 1
Theorem: If T is a tree with at least two vertices, then it has at least 2 leaves. **Lemma:** Every tree is bipartite

9.6 Spanning Tree

A spanning tree of G is a spanning subgraph of G that is a tree. In other words $V(T) = V(G), E(T) \subseteq E(G)$, T is a tree.
Theorem: A graph G is connected iff G has a spanning tree.
Corollary: If G is a connected graph with n vertices and $n-1$ edges, then G is a tree.
Corollary: Let G be a graph with n vertices. If any of the following 3 condition holds, then G is a tree

- G is connected
- G has no cycles
- G has $n-1$ edges

9.7 Characterizing Bipartite Graphs Theorem

A graph is bipartite iff it has no odd cycles

9.8 Planarity

A planar embedding of a graph G is a drawing of the graph in the plane so that its edges i ntersect only at the ends (edges don't cross) and no two vertices occupy the same point. A graph with a planar embedding is called a planar.

9.9 Face

A face of a planar embedding is an undivided region of the plane
The **boundary** of a face is the subgraph formed from all vertices and edges that touch the face.
Two faces are **adjacent** if their boundaries have at least one edge in common.
For a planar embedding of a connected graph, the boundary walk of a face is a closed walk once around the perimeter of the face boundary. The degree of a face f is the length of the boundary walk of the face, denoted $deg(f)$.

9.10 Handshake Lemma for Faces

Let G be a planar graph with a planar embedding where F is the set of all faces. Then

$$\sum_{f \in F} deg(f) = 2|E(G)|$$

Lemma: In a planar embedding, an edge is a bridge iff the two sides of e are in the same face
Jordan Curve Theorem: Every planar embedding of a cycle separates the plane into two regions, one on the inside and one on the outside.

9.11 Euler's Formula

Let G be a connected graph with n vertices and m edges. Every planar embedding of G has f faces, where $n - m + f = 2$

9.12 Platonic Graph

A graph is platonic if it has a planar embedding in which every vertex has degree $d \geq 3$ and every face has degree $d^* \geq 3$.
Theorem: There are exactly 5 platonic graphs.

9.13 Nonplanar Graph

Lemma: Let G be a planar graph with n vertices and m edges. If there is a planar embedding of G where every face has degree at least $d \geq 3$, then $m \leq \dfrac{d(n-2)}{d-2}$
Lemma: Let G be a planar graph with $n \geq 3$ vertices and m edges. Then $m \leq 3n - 6$.
Lemma: If G contains a cycle, then in any planar embedding of G, every face boundary contains a cycle.
Corollary: $K_5, K_{3,3}$ is not planar.
Theorem: Let G be a bipartite planar graph with $n \geq 3$ vertices and m edges. Then $m \leq 2n - 4$

9.14 Edge Subdivision

An edge subdivision of G is obtained by replacing each edge of G with a new path of length at least 1.
fact: A graph is planar iff any edge subdivision of the graph is planar
Corollary: If G contains an edge subdivision of $K_{3,3}, K_5$ as a subgraph, then G is not planar.

9.15 Kuratowski's Theorem

A graph is planar iff it does not contain an edge subdivision of $K_{3,3}, K_5$ as a subgraph.

10 Week 10

10.1 Colouring

A k colouring of a graph G is an assignment of a colour to each vertex using one of k colours, so that adjacent vertices have different colours. More precisely, if C is a set of size k, then a k-colouring is a function $f : V(G) \to C$ such that $f(u) \neq f(v)$ for all $uv \in E(G)$. A graph that has $k-$coloruing is called k-colourable.

10.2 Colour Theorem

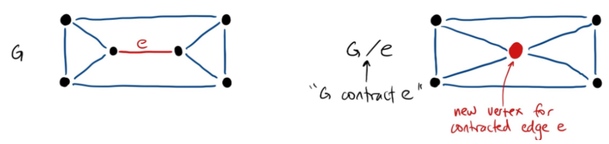
Theorem: A graph is 2-colourable iff it is bipartite
Theorem: The complete graph K_n is n-colourable and not k-colourable for any $k < n$
Lemma: Every planar graph has at least 1 vertex of degree at most 5

10.3 4/5/6-colour theorem

Every planar graph is 4/5/6-colourable.

10.4 Contracting

Let e be an edge in graph G. The graph G/e formed by contracting $e = uv$ removes e and squeezes the two ends of e into one vertex, preserving all edges incident with either end.



10.5 Matching

A matching of a graph is a set of edges in which no two edges share a common vertex.

10.6 Saturated

In a matching M, a vertex is saturated by M if v is incident with an edge in M.

10.7 Perfect matching

A matching is a perect matching if it saturates every vertex

10.8 Cover

A cover of a graph G is a set of vetices C such that every edge of G has at least one endpoint in C.

10.9 Cover Lemmas

Lemma: If M is a matching of G and C is a cover of G then $|M| \leq |C|$
Lemma: If M is a matching and C is a cover and $|M| = |C|$, then M is a maximum matching and C is a minimum cover.

11 Week 11

11.1 Alternating / Augmenting Path

An alternating path P with respect to a matching M is a path where consecutive edges alternate between being in M and not in M. An augmenting path is an alternating path that starts and ends with distinct unsaturated vertices.

11.2 Lemma

If a matching M has an augmenting path, then M is not maximum.

11.3 Konig's Theorem

In a bipartite graph, the size of a maximum matching is equal to the size of a minimum cover.

11.4 The Bipartite Matching Algorithm (X-Y Construction)

Given a bipartite graph G with bipartition (A, B) and a matching M of G.

1. Let X_0 be the set of all unsaturated vertices in A
2. Set $X \leftarrow X_0, Y \leftarrow \varnothing$
3. Let N be the set of all neighbours of X in B not currently in Y
    - If at least one vertex $v \in N$ is unsaturated, then we have found an augmenting path. Make a larger matching by swapping edges in the augmenting path. Then start over step 1
    - If all vertices in N are saturated then put all of them in Y. Add their matching neighbours to X. Go to step 3
    - IF no such neighbour vertices ($|N| = 0$), then stop. The matching is maximum, and the minimum cover is $Y \cup (A \backslash X)$

Corollary: Let G be a bipartite graph on m edges with bipartition (A, B) such that $|A| = |B| = n$. Then G has a matching size of at least m/n.

11.5 Hall's Theorem

A bipartite graph G with bipartition (A, B) has a matching saturating every vertex in A iff every subset $D \subseteq$ A satisfies $|N(D)| \geq |D|$

11.6 Corollary

A bipartite graph G with bipartition (A, B) has a perfect matching if and only if $|A| = |B|$ and every subset $D \subseteq A$ satisfies $|N(D)| \geq |D|$

11.7 Theorem

If G is a $k-$regular bipartite graph with $k \geq 1$, then G has a perfect matching

11.8 Edge-colourings

An edge k-colouring of a graph G assigns one of k colors to each edge of G such that two edges incident with the same vertex are assigned different colors.

11.9 Theorem

Every bipartite graph with maximum degree Δ has an edge Δ-colouring

11.10 Lemma

Let G be a bipartite graph having at least one edge. Then G has a matching saturating every vertex of maximum degree.