MATH 138 - Formulas and Theorems

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1 Week 1

1.1 Partition

A partition P_n for interval $[a, b]$ is a finite sequence of increasing number of the form

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

Partition subdivides [a,b] into n subintervals (does NOT need to be regular)

$$[t_0, t_1], [t_1, t_2], \ldots, [t_{n-1}, t_n]$$

1.2 Riemann Sum

Given a **BOUNDED** function f on [a,b] a partition P_n on [a,b] and a set of $\{c_1, c_2, \ldots, c_n\}$ where $c_i \in [t_{i-1}, t_i]$ then a Riemann Sum for f with respect to P is given by

$$S_n = \sum_{i=1}^{n} f(c_i)\Delta t_i$$

1.3 Integrability

A function is **integrable** on [a,b] if there exists a **unique** number $I \in \mathbb{R}$ s.t. for any sequence of partition $\{P_n\}$ with $\lim_{n\to\infty} \|P_n\| = 0$ and any sequence of Riemann Sums $\{S_n\}$ associated with P_n we have

$$\lim_{n\to\infty} S_n = I$$

Denoted by integral of f over $[a, b]$ and denoted by

$$\int_a^b f(x)dx = \lim_{n\to\infty} S_n = I$$

Note: Can also be true if there are a **FINITE** amount of discontinuities

1.4 Integrability Theorem

If f is continuous on $[a, b]$ then F is **integrable** on $[a, b]$

1.5 Regular Partition

Subintervals with the same length denoted by

$$\Delta t = \frac{b-a}{n} \text{ and } t_i = t_0 + i\Delta t$$

1.6 Right Hand Riemann Sum

$$R_n = \sum_{i=1}^{n} f(t_i)\Delta t$$

Equivalent definition for left-hand and midpoint Riemann Sum $t_{i-1}, t_{i-1/2}$. And if f is **integrable** then,

$$\int_a^b f(x)dx = \lim_{n\to\infty} R_n$$

1.7 Properties

- $\forall c \in \mathbb{R}, \int_a^b cf(x)dx = c\int_a^b f(x)dx$
- $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
- $\forall x \in [a,b], m \le f(x) \le M \implies m(b-a) \le \int_a^b f(x)dx \le M(b-a)$
- $f(x) \ge 0 \implies \int_a^b f(x)dx \ge 0$
- $f(x) \ge 0 \implies \int_a^b f(x)dx \ge 0$
- $f(x) \le g(x) \implies \int_a^b f(x)dx \le \int_a^b g(x)dx$
- $|f(x)|$ is integrable on [a,b], and $|\int_a^b f(x)dx| \le \int_a^b |f(x)|dx$
- $f(a)$ is defined $\implies \int_a^a f(x)dx = 0$
- f is integrable on $[a,b] \implies \int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\forall a,b,c \in I, f$ is integrable on $I \implies \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

1.8 Average Value

If f is continuous on $[a,b]$, the average value of f on $[a,b]$ is defined as

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$$

1.9 Average Value Theorem

If f is **continuous** on $[a,b]$. There exists $c \in [a,b]$ s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

2 Week 2

2.1 FTC I

If f is **continuous** on an **open** interval I containing $x = a$ and if

$$G(x) = \int_a^x f(t)dt$$

then G is differentiable for all $x \in I$ and $G'(x) = f(x)$ s.t.

$$G'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

2.2 Indefinite Integral

The collection for all antiderivative of $f(x)$ is denoted by

$$\int f(x)dx = F(x) + c$$

where $c \in \mathbb{R}$ and F is an antiderivative.

2.3 Common anti-derivative

$\int x^n dx$	$\frac{x^{n+1}}{n+1} + c$ (for $n \neq -1$)
$\int \frac{1}{x}\, dx$	$\ln\|x\| + c$
$\int e^x\, dx$	$e^x + c$
$\int \sin(x)\, dx$	$-\cos(x) + c$
$\int \cos(x)\, dx$	$\sin(x) + c$
$\int \sec^2(x)\, dx$	$\tan(x) + c$
$\int \frac{1}{1+x^2}\, dx$	$\arctan(x) + c$
$\int \frac{1}{\sqrt{1-x^2}}\, dx$	$\arcsin(x) + c$
$\int -\frac{1}{\sqrt{1-x^2}}\, dx$	$\arccos(x) + c$
$\int \sec(x)\tan(x)\, dx$	$\sec(x) + c$
$\int a^x\, dx$	$\frac{a^x}{\ln(a)} + c$ (for $a > 0$)

2.4 FTC II

If f is **continuous** on [a,b] and F is any antiderivative of f then,

$$\int_a^b f(x)dx = F(b) - f(a) = [F(x)]_a^b$$

2.5 Corollary: Extended Version of FTC

If f is **continuous** and g and h are differentiable then,

$$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)dt\right] = f(h(x))h'(x) - f(g(x))g'(x)$$

2.6 Change of variable / Substitution

If $g'(x)$ is **continuous** on $[a, b]$ and f(x) is **ALSO CONTINUOUS** between $g(a)$ and $g(b)$ then,

$$\int_{x=a}^{x=b} f(g(x))g'(x)dx = \int_{u=g(a)}^{u=g(b)} f(u)du$$

3 Week 3

3.1 Trig Substitution

- Remember to add the a^2 to the x sub
- **Recalculate lower bound and upperbound**

3.2 Integration by Parts

3.3 Partial Fractions

4 Week 4

4.1 Type I Improper Integral

Let f be integrable on $[a, b]$ for all $a \leq b$.

$$\int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)dx$$

$$\int_{a}^{\infty} f(x)dx = \lim_{b \to \infty} \int_{a}^{b} f(x)dx$$

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a \to -\infty} \int_{a}^{0} f(x)dx + \lim_{b \to \infty} \int_{0}^{b} f(x)dx$$

The integrals converge if all of the limits exist. It diverges even if **ONE limit is DNE**.

4.2 p-test for Integrals

$$\int_{1}^{\infty} \frac{1}{x^p} dx \iff p > 1$$

$$p > 1 \implies \int_{1}^{\infty} \frac{1}{x^p} dx = \frac{1}{p-1}$$

4.3 Properties of Type I improper Integrals

Suppose that $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ converge.

$$\int_{a}^{\infty} cf(x)dx \text{ converges for any } c \in \mathbb{R} \implies \int_{a}^{\infty} cf(x)dx = c\int_{a}^{\infty} f(x)dx$$

$$\int_{a}^{\infty} f(x) + g(x)dx \text{ converges } \implies \int_{a}^{\infty} f(x)g(x)dx = \int_{a}^{\infty} f(x)dx + \int_{a}^{\infty} g(x)dx$$

$$\forall x \geq a, f(x) \leq g(x) \implies \int_{a}^{\infty} f(x)dx \leq \int_{a}^{\infty} g(x)dx$$

$$a < c < \infty \implies \int_{a}^{\infty} f(x) \text{ converges and } \int_{a}^{\infty} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

4.4 Comparison Test for Type I Improper Integrals

Suppose f and g are continuous functions where $f(x) \geq g(x) \geq 0$ for $x \geq a$
1) If $\int_a^\infty f(x)dx$ converges, $\int_a^\infty g(x)dx$ converges too
2) If $\int_a^\infty g(x)dx$ diverges, $\int_a^\infty f(x)dx$ diverges too
Note: $f(x) \geq g(x) \geq 0$

4.5 Absolute Convergence Theorem (ACT)

Let f be integrable on [a,b] for all $b \geq a$. Then $|f|$ is integrable on $[a, b]$ for all $b \geq a$ and if $\int_a^\infty |f(x)|$ converges, so does $\int_a^\infty f(x)dx$.
Note: if $|f(x)| \leq g(x)$ for $x \geq a$ and if $\int_a^\infty g(x)dx$ converges so does $\int_a^\infty f(x)dx$

4.6 Type II Improper Integral

Consider $\int_a^b f(x)dx$

If there is a discont at $x = a$ then we use $\lim_{t\to a^+} \int_t^b f(x)dx$

If there is a discont at $x = b$ then we use $\lim_{t\to b^-} \int_a^t f(x)dx$

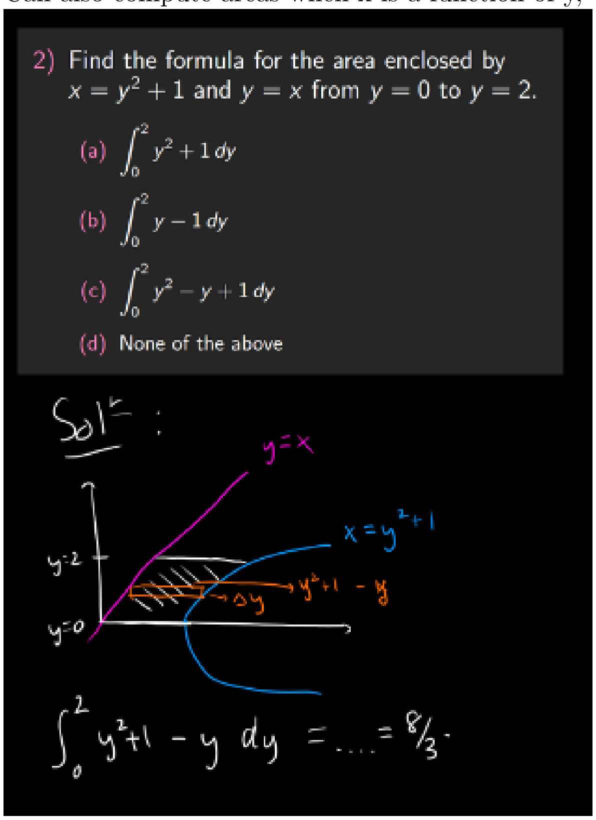
If f is not continuous at some $c \in (a,b)$ then we can split it up into $\int_a^c f(x)dx + \int_c^b f(x)dx$ and reconsider the previous cases

4.7 Area Between Curves

$$\int_a^b |f(x) - g(x)|dx$$

"Upper" - "Lower"

Can also compute areas when x is a function of y, "outer" - "inner" or "right" - "left"



5 Week 5

5.1 Disk / Washer General Formula

$$V = \int_a^b \pi(f(x)^2 - g(x)^2)dx$$

Note: if rotating around a horizontal line that's not the x-axis, $(a - f(x))$ or $(f(x) - a)$

5.2 Cylindrical Shells General Formula

$$V = \int_a^b 2\pi x(f(x) - g(x))dx$$

Note: if rotating around a vertical line that's not the y-axis, $|x - a|$

5.3 Ordinary Differential Equation (ODE)

An equation containing derivatives of a dependent variable $y = f(x)$

$$F(x, y, y', y'', \dots, y^{(n)}) = 0$$

5.4 Order of ODE

The highest derivative that appears e.g. $y' + y'' + y''' = 0$, order = 3

5.5 Lineararity of ODE

An ODE is linear if it contains only linear functions in y, y', y''

5.6 General Solution of ODE

Collection of all possible solutions including arbitrary constant

5.7 Particular Solution

A solution in which all arbitrary constants have been determined. Initial value problem (IVP) is an ODE that comes with initial conditions.

5.8 Direction field

Consider the ODE

$$\frac{dy}{dx} = f(x, y)$$

It tells us how the slope of the tangent line behaves at each point (x, y)

6 Week MT

6.1 Separable ODE

It is a **first order** ODE that is written as

$$\frac{dy}{dx} = g(y)h(x)$$

That is we can factor the right-hand side into a product of functions into

$$\int \frac{1}{g(y)} dy = \int h(x) dx$$

6.2 Substitution Method

Sometimes, it is not separable, but substitution makes it separable.
Common substitutions: $v = y + x, v = \frac{y}{x}, v = y'$

6.3 Linear first-order ODE

General form is

$$A(x)\frac{dy}{dx} + B(x)y + C(x), A(x) \neq 0$$

By dividing $A(x)$, we can write it as

$$\frac{dy}{dx} + P(x)y = Q(x)$$

6.4 Algorithm to Solve linear first-order ODE

1. Write ODE in form of $\frac{dy}{dx} + P(x)y = Q(x)$
2. Find $\mu(x) = e^{\int P(x)dx}$
3. Multiply the ODE by $\mu(x)$ to collapse LHS into $\frac{d}{dx}(\mu(x)y)$
4. Integrate both sides and solve for y

General Formula to ODE $\frac{dy}{dx} + P(x)y = Q(x)$

$$y = \frac{1}{\mu(x)} \left(\int \mu(x)Q(x)dx \right), \mu(x) = e^{\int P(x)dx}$$

6.5 Existence and Uniqueness Theorem for First-Order Linear Differential Equations

Assume P and Q are **continuous** functions on an interval I. Then for each $x_0 \in I$ and for all $y_0 \in \mathbb{R}$, the IVP

$$y' + P(x)y = Q(x)$$

$$y(x_0) = y_0$$

has exactly one solution $y = \gamma(x)$ on interval I

— **MID TERM CUTOFF** —

6.6 Newton's Law of Cooling

T is the object's temperature, T_a is the ambient temperature

$$\frac{dT}{dt} = -k(T - T_a), k > 0$$

6.7 Natural growth

$$\frac{dP}{dt} = kP, k > 0$$

6.8 Logistic growth

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), k, M > 0$$

where M is the carrying capacity

7 Week 6

7.1 Infinite series

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. An infinite series is an expression of the form

$$a_1 + a_2 + a_3 + a_4 + \cdots = \sum_{n=1}^{\infty} a_n$$

7.2 Sequence of partial sums

If $\sum_{n=1}^{\infty} a_n$ is a series, then the sequence of partial sums $\{S_n\}$

$$S_n = a_1 + a_2 + \cdots + a_n$$

7.3 Convergence and Divergence

A series $\sum_{n=1}^{\infty} a_n$ converges to $S \in \mathbb{R}$ if $\lim_{n\to\infty} S_n = S$ S is called the sum of the series. If $\{S_n\}$ diverges, we say the series **diverges**.

7.4 Geometric Series

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \ldots, r \in \mathbb{R}$$

7.5 Geometric Series Test

The geometric series $\sum_{n=0}^{\infty} r^n$ converges if $|r| < 1$ and diverges otherwise. If $|r| < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

7.6 Properties

Suppose $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ Then,

1. $\sum_{n=1}^{\infty} a_n = kA$
2. $\sum_{n=1}^{\infty} a_n \pm b_n = A \pm B$
3. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=j}^{\infty} a_n$ also converges for each $j \geq 1$
4. If $\sum_{n=j}^{\infty} a_n$ converges for some j, then $\sum_{n=1}^{\infty} a_n$ also converges.

(Note: also applies for starting with n = 0 for 3 and 4).

7.7 Divergence Test

If $\lim_{n\to\inf} a_n \neq 0$ (or DNE) then, $\sum_{n=1}^{\infty} a_n$ diverges.
(Contrapositive: If $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} = 0$)

8 Week 7

8.1 Series Positive

A series $\sum_{n=1}^{\infty} a_n$ is called **positive** if $a_n \geq 0$ for $n \in \mathbb{N}$

8.2 Integral Test

Suppose $f(x)$ is

- continuous
- positive
- decreasing

for $x \in [1, \infty)$ and let $a_n = f(n)$. Then,
$\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.

8.3 Theorem: p-series Test

The series $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges if and only if $p > 1$

8.4 Remainder

The remainder is the error in using S_n to approximate $\sum_{n=1}^{\infty} a_n = S$ so

$$R_n = S - S_n = a_{n+1} + an + 2 + \dots$$

If $a_n = f(n)$ and $f(x)$ is continuous, positive and decreasing, we know that

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$$

so we get an upper bound on the remainder.

8.5 Comparison Test

Suppose $0 \leq a_n \leq b_n$ for $n \in \mathbb{N}$ then,

1. $\sum b_n$ converges, then $\sum a_n$ converges too
2. $\sum a_n$ diverges, then $\sum b_n$ diverges too

8.6 Limit Comparison Test (LCT)

Suppose $a_n \geq 0$ and $b_n > 0$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} \dfrac{a_n}{b_n} = L$ Then,

1. $0 < L < \infty$: $\sum a_n$ and $\sum b_n$ both converges of both diverges.
2. $L = 0$: $\sum b_n$ converges then $\sum a_n$ converges
3. $L = 0$: $\sum a_n$ diverge then $\sum b_n$ diverge
4. $L = \infty$: $\sum a_n$ converges then $\sum b_n$ converges
5. $L = \infty$: $\sum b_n$ diverges then $\sum a_n$ diverges

8.7 Alternating Series

The terms are alternating positive and negative

8.8 Alternating Series Test (AST)

Suppose $a_n > 0$ for all $n \in N$ Consider $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ If

1. $\{a_n\}$ is non increasing (eventually) $a_n \geq a_{n+1}$
2. $\lim_{n\to\infty} a_n = 0$

Then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

8.9 Estimating Sums of Alt Series

- Check if series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is converging by AST
- This means the actual sum lies between any two consecutive partials sums so the error satisfies

$$|R_n| = |S - S_n| \leq |S_{n+1} - S_n| = |\pm a_{n+1}| = a_{n+1}$$

- That is the error in using S_n is bounded above $|R_n| \leq a_{n+1}$

9 Week 8

9.1 Absolute Convergence

A series converges if $\sum_{n=1}^{\infty} |a_n|$ converges. Functions that are ≥ 0 there have no differences.

9.2 Conditional Convergence

A series conditionally converges if it does not converge absolutely but converges.

9.3 Absolute Convergence Theorem (ACT)

If $\sum_{n=1}^{\infty} |a_n|$ converges, so does $\sum_{n=1}^{\infty} a_n$.
Note: it will converge to different values.

9.4 Steps to approach problem

1. Try Divergence Test
2. check for Absolute Convergence
3. check for Conditional Convergence

9.5 Rearrangement

Assume a series has sum S
If it absolutely converges, if we rearrange, it will have sum S.
If it conditionally converges, if we rearrange, it will have a different value. (Riemann Rearrangement Theorem)

9.6 Ratio Test

Check $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$. And $L \in \mathbb{R}$ or $L = \infty$

- $L < 1$ Absolutely Converges
- $L = 1$ Inconclusive
- $L > 1$ Diverges

9.7 Root Test

Check $\lim_{n\to\infty} \sqrt[n]{\left|\frac{a_{n+1}}{a_n}\right|} = L$. And $L \in \mathbb{R}$ or $L = \infty$

- $L < 1$ Absolutely Converges
- $L = 1$ Inconclusive
- $L > 1$ Diverges

10 Week 9

10.1 Power Series

$$\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots$$

10.2 Series Convergence

$\sum_{n=0}^{\infty} a_n(x-a)^n$ there are a few possibilities:

1. The series converges only when $x = a$ $(R = 0)$
2. The series converges for all $x \in R$ $(R = \infty)$
3. There exists $R \in (0, \infty)$ such that it converges absolutely for $|x-a| < R$, and diverges if $|x-a| > R$ and unknown if $|x-a| = R$.

10.3 Abel's Theorem

If $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ has an interval of convergence I then f is continuous on I.

10.4 Integral and Differentiation

If $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ with radius of convergence $R > 0$, then $f(x)$ is differentiable on $(a-R, a+R)$ and

1. $f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$
2. $\int f(x)dx = \sum_{n=0}^{\infty} \dfrac{a_n(x-a)^{n+1}}{n+1} + C$

Note: radius stays the same but interval of converge may change at endpoints.

10.5 Proposition

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

11 Week 10

11.1 n-th degree Taylor Polynomial centered at x = a

$$T_{n,a}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

11.2 Taylor's Theorem

Suppose f is (n + 1) times differentiable throughout an interval I containing a. For every $x \in I$, the error in approximating $f(x)$ with $T_{n,a}(x)$ has the form of

$$R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

11.3 Taylor's Inequality

Suppose f is (n+1) times differentiable throughout an interval I containing a. If $x \in I$ and $|f^{(n+1)}(c)| \leq M$ for all c between a and x.

$$|R_{n,a}(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$$

11.4 * Taylor Series

Assume f has derivatives of all order at $a \in \mathbb{R}$, we say the following is a taylor series centered at x = a,

$$f(x) \approx \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Note : Theorem assumes that f has a power series and it conclude thats it is a Taylor Series. Does NOT say every function is equal to its taylor series. Also, no matter how the series is found, you will get a Taylor's Series.

11.5 Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

11.6 Function equal to their Taylor series for all x

$$f(x) = \lim_{n\to\infty} T_{n,a}(x)$$

Since $f(x) = T_{n,a}(x) + R_{n,a}(x)$. Check

$$\lim_{n\to\infty} R_{n,a}(x) = 0$$

11.7 * Convergence of Taylor Series

Suppose f has derivatives of all orders of an interval I containing $x = a$. If there exists a constant $M \in \mathbb{R}$ with $|f(n)(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in I$ then,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \forall x \in I$$

11.8 Corollary

$\sin(x), \cos(x)$ are equal to their Taylor series for all $x \in (-\infty, \infty)$

12 Week FE

12.1 Binomial Theorem

Let $k \in \mathbb{N}$, then for all $x \in \mathbb{R}$

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n$$

where

$$\binom{k}{n} = \frac{k!}{n!(k-n)!} = \frac{k(k-1)\dots(k-n+1)}{n!}$$

12.2 Binomial Theorem as a Maclarin Series

$$R = 1, I \in (-1,1)$$

$$\forall x \in (-1,1), (1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n$$

12.3 Claims

Let $f(x) = \sum_{n=0}^{k} \binom{k}{n} x^n$

1. $\binom{k}{n+1}(n+1) + \binom{k}{n}n = \binom{k}{n}k$ for $n \geq 1$
2. $f'(x) + xf'(x) = kf(x)$ for all $x \in (-1,1)$
3. $\left(\frac{f(x)}{(1+x)^k}\right)' = 0$ for all $x \in (-1,1)$

12.4 Generalized Binomial Theorem

Let $k \in \mathbb{R}$, then for all $x \in (-1,1)$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

where

$$\binom{k}{n} = \frac{k!}{n!(k-n)!} = \frac{k(k-1)\dots(k-n+1)}{n!}, \binom{k}{0} = 1$$

13 Important Representations to remember

13.1 Series

- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} (R = \infty)$
- $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} (R = \infty)$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} (R = \infty)$
- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n (R = 1)$
- $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n (R = 1)$